Analysis of optimal velocity model with explicit delay

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(Received 15 May 1998)

We analyze the optimal velocity model (OVM) with explicit delay. The properties of congestion and the delay time of car motion are investigated by analytical and numerical methods. It is shown that the small explicit delay time has almost no effects. In the case of the large explicit delay time, a new phase of congestion pattern of OVM seems to appear. [S1063-651X(98)12410-8]

PACS number(s): 64.60.Cn, 64.60.Ht, 02.60.Cb, 05.70.Fh

I. INTRODUCTION

In recent years, we proposed a car-following model called the ''optimal velocity model'' (OVM), based on a dynamical equation [1]

$$\ddot{x}_n(t)=a\{V(x_{n+1}(t)-x_n(t))-\dot{x}_n(t)\}, \qquad (1.1)$$

where t is time and x_n is a position of the nth car. Cars are numbered so that the $(n+1)$th car precedes the nth car. The driver feels the headway $x_{n+1}(t)-x_n(t)$ and determines an optimal velocity $V(x_{n+1}(t)-x_n(t))$. It is best to drive a car with the optimal velocity but in general a deviation exists between the optimal velocity and a real one. The driver responds to the deviation $\Delta V=V(x_{n+1}(t)-x_n(t))-\dot{x}_n(t)$ and diminishes it by giving an acceleration $a\Delta V$ to the car. The coefficient a expresses the sensitivity of the driver. We call the function V the ''optimal velocity function'' (OVF). In previous papers, we have shown how the OVM can explain behaviors of traffic flow, for example, the transition from a free flow to a congested flow, a density-flow relationship, a kind of effective delay of car motion [1–4].

On the other hand, the prototype equation of motion of traditional car-following model is

$$\ddot{x}_n=\lambda_0\{\dot{x}_{n+1}-\dot{x}_n\}, \qquad (1.2)$$

where λ_0 is a constant [5–7]. In this model, a driver is thought to react to the stimulus proportional to the relative velocity between the previous car and his own car. Equation (1.2) may be generalized by changing the constant λ_0 to a function $\lambda(x_{n+1}-x_n)$ of headway. However, these models have no physically interesting solution because such equation can be integrated easily and be reduced to the following equation:

$$\dot{x}_n=V(x_{n+1}-x_n), \qquad (1.3)$$

where V is a function of headway and $V'(x_{n+1}-x_n)=\lambda(x_{n+1}-x_n)$. In car-following models, therefore, the introduction of ''delay'' is necessary and is essential to understanding traffic dynamics [8,9]. The following equation is a typical one that is widely used in car-following models:

$$\dot{x}_n(t+\tau)=V(x_{n+1}(t)-x_n(t)), \qquad (1.4)$$

where τ is a delay time of the driver's response. The driver senses headway at time t and changes the velocity of his car at later time $t+\tau$. This delay time τ of response has been thought to be inevitable because it comes from the driver's physical delay of response to the stimulus together with the mechanical response time of a car. In this paper, this τ will be called the ''explicit delay time.''

The notion of explicit delay time τ is completely different from that of the ''delay time of car motion'' introduced in our previous paper [4]. Let us recall the definition of the delay time of car motion. Consider a pair of cars, a leader and a follower. Assume the leader changes the velocity according to $v_l=v_0(t)$ and the follower duplicates the leader's velocity but with some delay time T, that is, $v_f=v_0(t-T)$. Under such a situation we can clearly define the delay time of car motion by T. It is known that the observed delay time T of car motion is of the order of 1 sec, but the known physical or mechanical response time τ is of the order of 0.1 sec. In the previous paper we confirmed that the Eq. (1.1) really produces T of order 1 sec.

We clarified that the OVM can describe the properties of traffic flows or the behaviors of cars fairly well without any explicit delay time τ. However, there exists, for a fact, the delay time of the response of the driver. The explicit delay time τ should be included in the dynamical equation in order to construct realistic models of traffic flow. It is a natural

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question what kind of effect appears in the traffic flow or in the car motion if we introduce the explicit delay in Eq. (1.1).

In this paper we investigate the following equation:

$$\ddot{x}_n(t+\tau)=a\{V(x_{n+1}(t)-x_n(t))-\dot{x}_n(t)\}. \quad (1.5)$$

In order for our analysis be more concrete, we use the parameter $a=2.0$ (1/sec) and the function V which are phenomenologically determined in previous papers by the observed data on Japanese motorways [10–12].

$$V(\Delta x)=16.8[\tanh 0.0860(\Delta x-25)+0.913], \quad (1.6)$$

in which the unit of length and time are meter and second, respectively.

The plan of this paper is as follows. In Sec. II we discuss the global properties of traffic flow in the OVM with the explicit delay. In Sec. III we investigate more microscopic property, that is, the delay time of car motion. First we discuss within a linear approximation and next evaluate the delay times of car motion in various cases by numerical simulations. As a special case, the car motion under the traffic signal is also treated. In Sec. IV we show the new feature of the OVM with the explicit delay. The final section is devoted to summary and discussion.

II. PROPERTY OF TRAFFIC FLOW IN OVM WITH EXPLICIT DELAY

A. Linear analysis

In this section we investigate the OVM with the explicit delay time τ of the driver's response described by Eq. (1.5).

First we analyze the linear stability of an N-car system on a circular lane of length L. Obviously, the homogeneous flow solution of Eq. (1.5) is given by

$$x_n^{(0)}(t)=V(b)t+nb, \quad b=L/N. \quad (2.1)$$

To see whether the solution (2.1) is stable or not, we add a small perturbation

$$x_n(t)=x_n^{(0)}(t)+y_n(t). \quad (2.2)$$

From Eq. (2.2) and Eq. (1.5), we can calculate a linearized equation with respect to $y_n(t)$

$$\ddot{y}_n(t+\tau)=a\{f\Delta y_n(t)-\dot{y}_n(t)\}, \quad (2.3)$$

where $f=V'(b)$ and $\Delta y_n=y_{n+1}-y_n$. A complete set of solutions is given by

$$y_{jn}(t)=\exp(i\alpha_j n+i\omega_j t), \quad (2.4)$$

where $\alpha_j=2\pi j/N$ for $j=1,2,3,\ldots,N$ and ω_j satisfies the equation

$$-\left(\frac{\omega_j}{a}\right)^2\exp\left[i\left(\frac{\omega_j}{a}\right)a\tau\right]=\left(\frac{f}{a}\right)\{e^{i\alpha_j}-1\}-i\left(\frac{\omega_j}{a}\right). \quad (2.5)$$

In Eq. (2.5), variables are combined to be dimensionless. The condition that each solution $y_{jn}(t)$ becomes marginally stable is Im $\omega_j=0$. For convenience of explanation, we will omit the mode index j and treat α as a continuous variable.

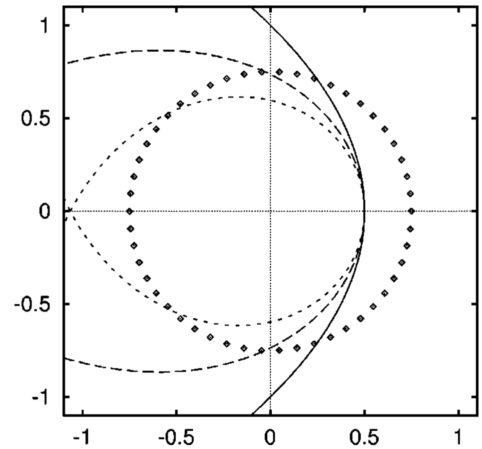

FIG. 1. Critical curves in the polar coordinate $(f/a,\alpha)$ plane. The solid line, dashed line, and dotted line show critical curves for $a\tau=0$, 0.2, and 0.4, respectively. A circle of diamond marks represents mode solutions for $f/a=0.75$.

The condition Im $\omega=0$ gives ''critical curves'' for each $a\tau$ in $(f/a,\alpha)$ plane, where f/a is a radial coordinate and α is an angular coordinate. Mode solutions $y_{jn}(t)$ are represented by a point $(f/a,\alpha_j)$ on a circle $f/a=$const.

Three critical curves for $a\tau=0$, 0.2, and 0.4 are shown in Fig. 1, in which a reference circle represents mode solutions for $f/a=0.75$. The modes staying outside (right-hand side) of the critical curve are unstable. Figure 1 shows that a homogeneous flow state with a parameter $f/a=0.75$ is an unstable state. From Fig. 1, it is found that unstable modes increase as the explicit delay time τ becomes large. This situation looks similar to a case where the sensitivity a becomes small in the original OVM [1]. There seems to be some relationship between the sensitivity a and the explicit delay time τ as indicated in Ref. [13].

B. Numerical simulations

The effect of the explicit delay in the congestion formation can be evaluated by numerical simulations. In previous papers [1,2], we investigated the property of traffic flows in a circuit. It is found that when the car density exceeds a critical value, a homogeneous traffic flow becomes unstable and makes a phase transition to a congested flow. After enough time, the congested flow becomes stationary and shows an alternating pattern of high-density (congestion cluster) and low-density regions. Each velocity and headway inside high-(low-) density regions always take common values that are determined only by the sensitivity a and OVF independently of any other conditions. The motion of each car can be shown in a ''phase space'' $(\Delta x,\ \dot{x})$, and the trajectories draw a single ''hysteresis loop,'' a kind of limit cycle. Figure 2 shows typical hysteresis loops for sensitivity $a=2.0$ and 2.8. Numerical simulations are carried out with the condition: total car number $N=100$ and circuit length $L=2500$ m [14]. These hysteresis loops are formed about $t\sim1000$ sec typically. As stated above, the results are independent of these conditions. Two turning points $C=(\Delta x_c,v_c)$ and $F=(\Delta x_f,v_f)$ correspond to the high- and low-density region for $a=2.0$ and C' and F' for $a=2.8$. We found the congestion pattern moves backward on the circuit with a constant

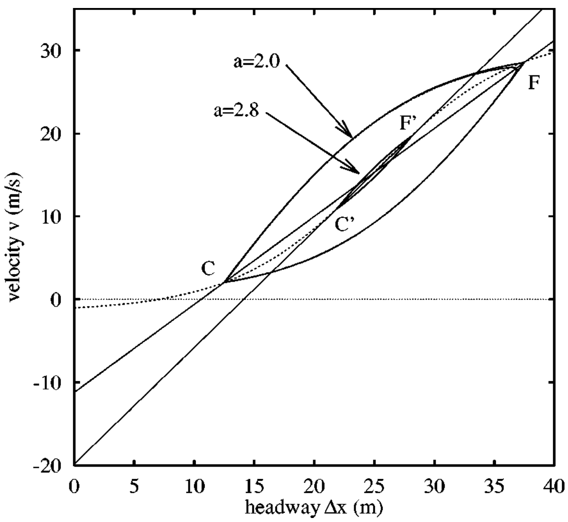

FIG. 2. ''Hysteresis loops'' for $a=2.0$ and $a=2.8$. Each line connects two turning points of each hysteresis loop. A tanh-type curve represents the OVF [Eq. (1.6)].

velocity $(v_f \Delta x_c - v_c \Delta x_f)/(\Delta x_f - \Delta x_c)$, which is given by the intersection of $\dot{x}$ axis and the line connecting two turning points C and F. Therefore the property of such congested flows is almost decided by two points C and F of hysteresis loop.

From numerical simulations, we recognize no qualitative difference in the behavior of the traffic flow between the cases with and without the explicit delay, if τ is not so large. Figure 3 shows hysteresis loops about $t \sim 5000$ sec for $\tau = 0$, 0.1, and 0.2, that is, $a\tau = 0$, 0.2, and 0.4. The changes of hysteresis loops are similar to those for the case that the sensitivity a becomes small in the original OVM [1]. Therefore it seems that the explicit delay time τ, which is not so large, does not play any essential role in the congestion formation. In other words, the effect of the explicit delay can be almost compensated by the change of sensitivity a.

Obviously, this is not the case for a very large $a\tau$. Figure 4 shows examples for $a\tau = 0.6$, 0.8, where critical curves are inside the referenced circle $f/a = 0.75$. In the original OVM instability always comes from long-range modes $(\alpha \sim 0)$, that is, short-range modes $(\alpha \sim \pi)$ are always stable. In the case $a\tau > 0.6$, however, there exist various cases in which all modes become unstable or short-range modes only become unstable. In such cases, the instability starts from all modes or from short-range modes. It is interesting to see what kind of phenomena emerge in such cases. An example shall be discussed in Sec. IV.

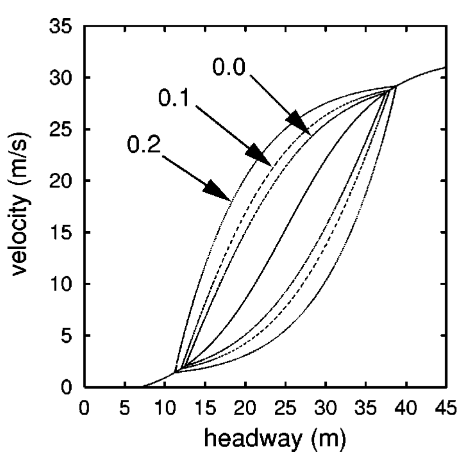

FIG. 3. Hysteresis loops for $\tau = 0$, 0.1, and 0.2. A tanh-type curve represents the OVF [Eq. (1.6)].

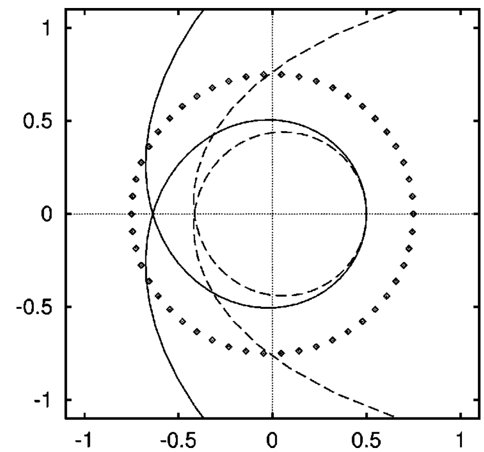

FIG. 4. Critical curves in the polar coordinate $(f/a, \alpha)$ plane. The solid line and dashed line show critical curves for $a\tau = 0.6$ and $a\tau = 0.8$, respectively. A circle of diamond marks represents mode solutions for $f/a = 0.75$.

III. DELAY TIME OF CAR MOTION

A. Linear analysis

In this section, we investigate the delay of car motion in order to see the effect of the explicit delay from a more microscopic viewpoint. First, we analyze the delay of car motion with the linear approximation.

Consider a pair of a leader and its follower where the leader moves with the velocity $v(t)$ and the follower replicates the motion of the leader after the time interval T, that is, the follower's velocity is given by $v(t-T)$. In this case we can define the delay time of car motion as T.

Let the position of the leader at time t be $y(t)$ and that of its follower $x(t)$, which obeys Eq. (1.5), that is,

$$\ddot{x}(t+\tau) = a\{V(y(t) - x(t)) - \dot{x}(t)\}. \tag{3.1}$$

Starting from the situation with headway b and velocity $V(b)$,

$$y_0(t) = V(b)t + b, \quad x_0(t) = V(b)t, \tag{3.2}$$

we investigate the response of the follower $\xi(t)$ to a small change $\lambda(t)$ of the leader:

$$y(t) = y_0(t) + \lambda(t), \quad x(t) = x_0(t) + \xi(t). \tag{3.3}$$

Inserting the above equations into Eq. (3.1) and taking a linear approximation, we get

$$\ddot{\xi}(t+\tau) + a\dot{\xi}(t) + af\xi(t) = af\lambda(t), \tag{3.4}$$

where $f = V'(b)$ is again a derivative of the OVF at headway b. If one takes $\lambda(t) = e^{i\omega t}$, the solution is given by

$$\xi(t)=\frac{1}{1+i\omega/f-e^{i\omega\tau}\omega^2/af}e^{i\omega t}. \tag{3.5}$$

This is rewritten as

$$\xi(t)=|\xi|e^{i\omega(t-T)}, \tag{3.6}$$

where

$$T=\frac{1}{\omega}\tan^{-1}\frac{a\omega-\omega^2\sin(\omega\tau)}{af-\omega^2\cos(\omega\tau)}, \tag{3.7}$$

$$|\xi|=\left[1+\left(\frac{\omega}{f}\right)^2-2\left(\frac{\omega^2}{af}\right)n\left(\cos\omega\tau+\frac{\omega}{f}\sin\omega\tau\right)+\left(\frac{\omega^2}{af}\right)^2\right]^{-(1/2)}. \tag{3.8}$$

First let us consider the case $|\omega|$ is sufficiently small ($\omega/f \ll 1, \omega/a \ll 1$). It will be discussed later whether this condition is satisfied or not in the realistic situation used in Eq. (1.6). Then we have

$$|\xi|=1,\quad T=\frac{1}{f}. \tag{3.9}$$

Here we take the general expression of $\lambda(t)$, which is expressed as follows:

$$\lambda(t)=\int\tilde{\lambda}(\omega)e^{i\omega t}d\omega. \tag{3.10}$$

$\tilde{\lambda}(\omega)$ is assumed to be nonzero only for ω small enough. Then we find that the follower's response becomes

$$\xi(t)=\int\tilde{\lambda}(\omega)e^{i\omega(t-T)}d\omega=\lambda(t-T), \tag{3.11}$$

that is,

$$\dot{x}(t)=V(b)+\dot{\xi}(t)=V(b)+\dot{\lambda}(t-T)=\dot{y}(t-T), \tag{3.12}$$

with T of Eq. (3.9).

As a result we conclude that for sufficiently slow and small change of leader's velocity, the delay time T of motion of the follower becomes $1/f$ (the inverse of derivative of the OVF at corresponding headway), independently of the explicit delay time τ of the driver's response.

B. Simulations for homogeneous flows

Next we will carry out numerical simulations to investigate the effect of the explicit delay in homogeneous traffic flows. The validity of the conditions $\omega \ll a,f$ can be checked also. We make simulations starting from homogeneous flows with various headways and add a small disturbance to a car. In order to set the homogeneous flow condition, the circuit length L is taken to be $N\times$ (headway). In this simulation also, we take $N=100$ for convenience. We suppose the disturbed car to be the first car. Therefore the 100th car precedes the first car. The delay time of car motion is estimated between the 10th car and 11th car when the disturbance propagates there.

TABLE I. Delay times of car motions in homogeneous flows.

Δx (m)	f^{-1} (s)	$T_{\tau=0}$ (s)	$T_{\tau=0.1}$ (s)	$T_{\tau=0.2}$ (s)
10	2.6427	2.6	2.6	2.6
15	1.3434	1.35	1.35	1.35
20	0.8282	0.95	0.95	0.95
25	0.6921	0.85	0.87	0.89
30	0.8282	0.95	0.95	0.95
35	1.3434	1.35	1.35	1.35
40	2.6427	2.6	2.6	2.6
50	13.101	13	13	13

In Table I, we summarize the results of numerical simulation. In the cases where the homogeneous flow is stable, the delay time T of car motion is almost equal to $1/f$ and the explicit delay has no effect. The cases $\Delta x=20,25,30$ correspond to the unstable situation. The measurement of the delay time T is carried out before the disturbance becomes large. The results show that the assumption $\omega \ll a,f$ is not valid. Even in such cases the explicit delay does not affect T.

C. Simulation for congested flows

In this subsection, we treat the car motion in a stationary congested flow, where linear analysis is no more valid obviously. In the previous paper [4] we have shown that the delay time T of car motion is the inverse of the gradient of lines that connect two turning points (C and F in Fig. 2). This is a natural extension of the statement obtained by the linear analysis: ''The delay time of car motion is the inverse of derivative of the OVF at corresponding headway.''

Our task here is to carry out similar numerical simulations with the explicit delay. After the congestion pattern becomes stable, all cars behave in the same manner expressed in Fig. 3. We can estimate the delay time T from the time interval of the motion of two successive cars, which is equivalent to the gradient of line connecting two turning points of the hysteresis loop. Table II shows the results of simulations for $\tau=0$, 0.1, 0.2.

The table clearly shows that the change of T is rather small compared to the change of τ. Therefore the main contribution of the delay of car motion comes from the structure of OVM itself and not from the explicit delay. The τ dependence of T appears only through the change of turning points of the hysteresis loop. In other words, the effect of the explicit delay is similar to the change of the sensitivity a and is not essential in the same as the previous section.

TABLE II. Delay times of car motions in congested flows.

τ	T simulation
0.0	0.94
0.1	0.96
0.2	0.99

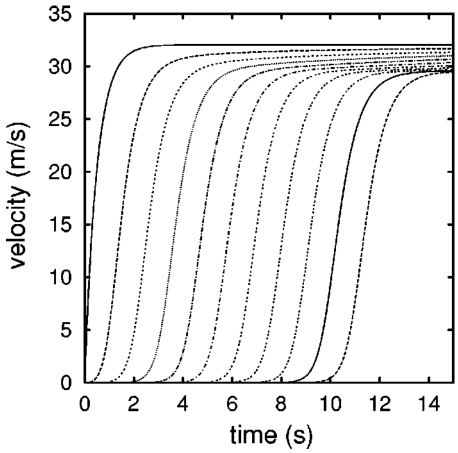

FIG. 5. Motions of cars 1–11 for $\tau=0$. Each curve shows the velocity of each car.

D. Simulations for car motion under a traffic signal

In this subsection we study the delay of motion of cars starting from a traffic signal. Though this may be a special case compared to previous subsections, experiments to observe the delay time have often been done in this situation.

Numerical simulations are carried out as follows. First a traffic signal is red and all cars are waiting with a headway of 7 m, at which the OVF (1.6) becomes zero. At time $t=0$, the signal changes to green and cars start.

Figures 5 and 6 show the velocities of several cars in a queue for the cases of $\tau=0$ and $\tau=0.2$ sec, respectively. It can be seen that cars with large car number (seventh or more) behave almost in the same manner as its preceding car.

We can estimate the delay time T from the behavior of the velocities of the 7th–10th cars. Table III shows the delay time of car motion for various τ. Again we find that the delay time T has a small dependence on the explicit delay time τ. To see whether this is general or not, we carried out another simulation with the initial headway, 3 m. For this purpose, the OVF [Eq. (1.6)] is changed to take zero for $\Delta x<7$ m. We show the results in the third column of Table III, which again show obviously a small dependence of T on τ.

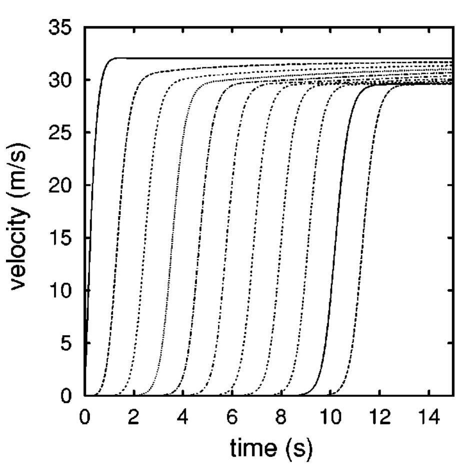

FIG. 6. Motions of cars 1–11 for $\tau=0.2$. Each curve shows the velocity of each car.

TABLE III. Delay times of car motions in queues starting from a traffic signal.

τ (s)	T for a 7-m headway (s)	T for a 3-m headway (s)
0.0	1.10	1.26
0.1	1.10	1.26
0.2	1.11	1.25
0.3	1.12	1.26

Hitherto we concerned the definition of delay time of car motion given in Sec. I: if velocities of two successive cars are given by $v(t)$ and $v(t-T)$ respectively, the delay time of car motion is T. This definition is valid only for the case that the motions of two cars are similar. As is seen from Figs. 5 and 6, the first several cars move in the different manner, because the headway of the first car is infinite but that of other cars are relatively small. In order to explore the delay time of car motion in such a case we will propose another definition. For example, we can define the delay time as the interval between the time when the preceding car starts and the time when the next car starts. Though there are many other possibilities, the above definition looks rather natural.

Figure 7 shows the delay time of car motion by the new definition. Obviously the data approach to a certain value as the car number becomes large. The limits of the delay times in this definition are the same values as those in the previous definition. It should be mentioned that the explicit delay time τ is simply added to the delay time T of car motion for first a few cars. This effect dissipates after several cars start.

From these results, we can conclude that the explicit delay time τ contributes directly to the delay time of car motion only for such a restricted case as for the motion of first a few cars starting from the traffic signal. In general case, the contribution of τ to T is rather small and is similar to the contribution from the change of the sensitivity a.

IV. NEW FEATURES OF OVM WITH EXPLICIT DELAY

In this section we show new features which exist only in the OVM with the explicit delay.

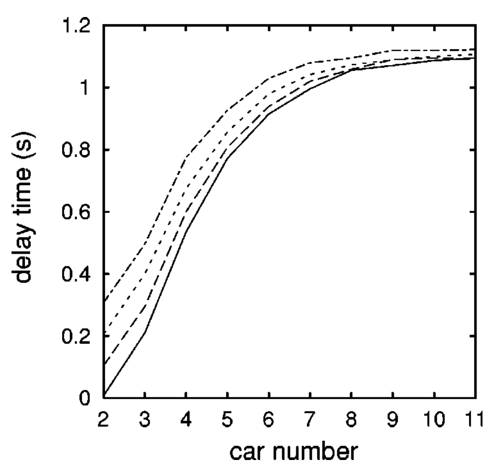

FIG. 7. Solid line connects delay times (time intervals) of 2nd–11th cars for $\tau=0$. Dashed, dotted, and dashed-dotted lines connect those for $\tau=0.1$, 0.2, and 0.3, respectively.

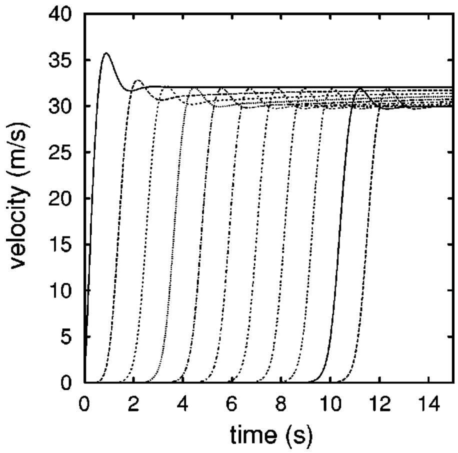

FIG. 8. Motions of cars 1–11 for $\tau=0.3$. Each curve shows the velocity of each car.

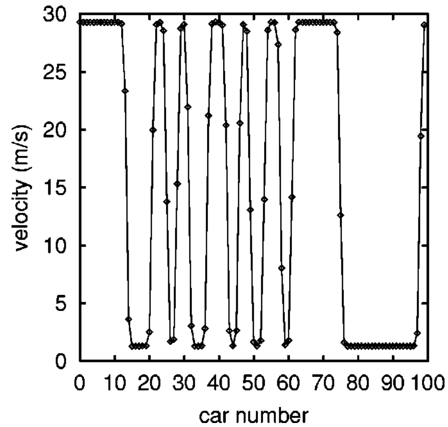

FIG. 9. A snapshot of velocities at $t=10\,000$ sec. Diamond marks represent the velocities of the cars.

A. Overshoot phenomenon

We investigated the motion of cars controlled by a traffic signal in the previous section. For small τ the motions of cars are not so different from those for $\tau=0$. For large τ, however, we can see a transitional overshoot of velocity, that is, an excess and a gradual decrease of velocity. As a typical case, the motions of cars for $\tau=0.3$ are shown in Fig. 8. We have carried out many numerical simulations by changing τ and found that the overshoot phenomenon begins at $\tau=0.19$ sec.

B. Upper bound of τ

First we note that the explicit delay time τ is understood as the summarized effect coming from delays of physical and mechanical response. Therefore a too large value will not be permitted from observations. There exists, however, more restrictive bound, which has an origin in the equation of motion (1.1) of the OVM.

We consider a homogeneous equation of the linearized equation (3.4) in the leader-follower system:

$$\ddot{\xi}(t+\tau)+a\dot{\xi}(t)+af\xi(t)=0. \tag{4.1}$$

$\xi(t)$ gives a perturbative motion of the follower when the leader moves in a constant velocity. In order that the two body system is stable, $\xi(t)$ must vanish as time develops. We see that $\xi(t)=e^{i\omega t}$ is a solution of Eq. (4.1), with ω satisfying

$$-\omega^2 e^{i\omega\tau}+ia\omega+af=0. \tag{4.2}$$

The marginally stable condition Im $\omega=0$ becomes

$$a\tau=\kappa\sin(\kappa),\quad f\tau=\kappa\cot(\kappa), \tag{4.3}$$

where $\kappa\equiv\mathrm{Re}\,\omega\tau$. By eliminating κ, we can find the upper bound of τ for given a and f. Though we could not solve Eq. (4.3) analytically, the upper bound τ_m is found to be a monotonic decreasing function of both a and f.

The value of τ_m can be evaluated numerically. For the sensitivity $a=2.0\,\mathrm{sec}^{-1}$ and the maximum value of $f=1.441\,\mathrm{sec}^{-1}$, which is read off from the OVF [Eq. (1.6)], the corresponding upper bound τ_m is 0.44 sec. If $\tau>\tau_m$ sec in the OVM with above a and f, the car cannot follow the constant velocity motion of the leader. Thus τ_m should be understood as the upper bound of the explicit delay time in order that the OVM is meaningful as a model of traffic flow.

C. New congestion pattern

Inside the above upper bound of the explicit delay time, some curious phenomena emerge in traffic flow as the explicit delay time becomes large. If such phenomena should be regarded as unrealistic, the upper bound will be taken at a smaller value.

Figure 9 shows a snapshot of headway at $t=10\,000$ sec, which is enough simulation time to settle congestion patterns. The conditions of the simulation are as follows: total car number $N=100$, circuit length $L=2500$ m and explicit delay time $\tau=0.22$ sec. There we can see small congestion clusters or rapid change of velocity between the 15th and 60th cars. This pattern looks like an intermediate pattern before the congestion is formed completely. However, in contrast to the case of $\tau=0.20$ sec, where such a pattern of small congestions disappears as time goes, the pattern has very long life and may never disappear in the case of $\tau=0.22$ sec. This pattern occupies a larger region as τ increases.

Next we take τ to be a larger value 0.4 sec. Figure 10 shows the hysteresis loops for $\tau=0$ at approximately $t\sim1000$ sec and 0.4 at approximately $t\sim10^6$ sec. Here we note that the OVF [Eq. (1.6)] takes negative value continuously for $\Delta x<7$ m and therefore cars can move backward (without collisions). Because such behaviors of vehicles are obviously unrealistic, it seems natural to set the upper bound of τ to the transition point at which this hysteresis loop appears.

As shown in Fig. 10, the profiles of hysteresis loops are qualitatively different. Moreover, the hysteresis loop for $a=2.8$ is larger than that for $a=2.0$ in the case of $\tau=0.4$ in contrast to the case of $\tau=0$. We also note that the relaxation time for $\tau=0.4$ is of the order of 10^3-10^4 times that for $\tau<0.2$. The differences of hysteresis loops and relaxation times seem to suggest an existence of a new phase. However, there exists another possibility: the stationary state indicated by this hysteresis loop is artificial due to finite size effects and a new phase does not exist. The congestion pattern changes continuously around $\tau\sim0.22$ sec and we cannot find

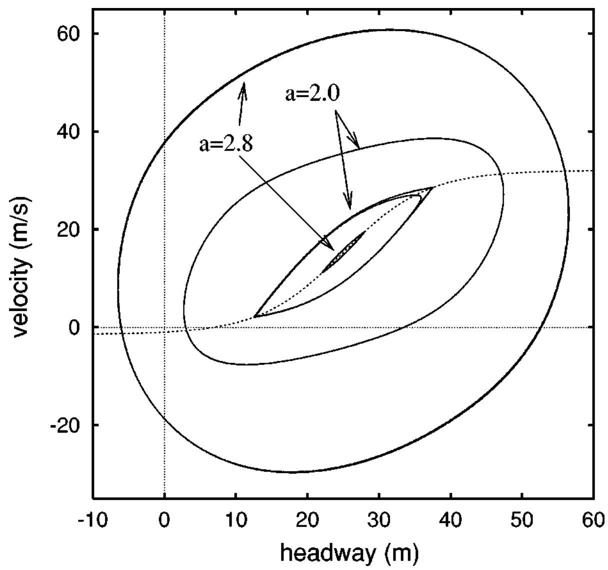

FIG. 10. Hysteresis loops for $\tau=0$ and $\tau=0.4$. As a reference, two cases, $a=2.0$ and $a=2.8$, are shown. A tanh-type curve represents the OVF [Eq. (1.6)].

a definite transition point. It is left to future work to determine whether this pattern indicates the existence of new phase or not.

V. SUMMARY AND DISCUSSION

In this paper we investigated the properties of the OVM with the explicit delay of the driver's response. The effects of the explicit delay are very small, if the delay time is small: $\tau<0.2$ sec. The effects are similar to the change (reduction) of the sensitivity a, and therefore the explicit delay does not play an essential role. This fact should be compared to the traditional car-following models, in which the delay of driver's response has played a significant role. The equation of motion of the traditional car-following model becomes trivial, if the delay time is zero.

For large explicit delay time τ, the traffic flow behaves in a different manner. If $\tau<0.2$ sec, the properties of congestion clusters are similar to that for $\tau=0$. For $\tau>0.2$ sec, the stationary pattern of the traffic flow does not consist of only such congestion clusters but confused patterns. For $\tau>0.3$ sec, the traffic flow becomes stationary but congestion clusters are never formed.

In the OVM, there is an upper bound of the explicit delay time, which comes from the condition that the equation of motion is meaningful. The upper bound, however, becomes small, if we require the existence of stable congestion clusters.

From this work, we can obtain an indication on a phenomenological study. In this paper we clarified the notions of the delay time τ of driver's response and the delay time T of car motion. However, the meaning of the delay time of response and its effect are model dependent. In traditional car-following models, the delay time τ seems to be merely a fitting parameter and so we can take any value for τ. Moreover, the delay time often takes different value in each term. In the OVM, the delay time τ is not free and the observed value decided by experiments will give a criterion whether the OVM with the optimal velocity function (1.6) is valid or not. Here we note that the contribution of the delay of driver's response to the delay of car motion is very small. The delay of car motion, therefore, has its root just in the dynamical equation itself. This fact suggests the difficulty in determining the delay time τ of driver's response by measuring the delay time T of car motion. Therefore τ must be measured directly by other experiments.

ACKNOWLEDGMENT

The authors are grateful to Y. Sugiyama for helpful discussions.

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